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A SIMPLE ONLINE FAIR DIVISION PROBLEM

arXiv:1903.10361

ONLINE FAIR DIVISION PROBLEMS

- **Objects arrive sequentially and to be allocated on the spot**

allocating profitable jobs (Uber), resources in cloud computing, food in a foodbank, tasks within a firm, refugees to localities

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OUR QUESTION:

A. optimal rules: **Welfare maximization** under the condition of **Fairness on average**

B. **dependence** on the **information** available to the rule

COMPARING TO THE LITERATURE

Economics. Welfare implications of congestion, signalling, and strategizing on dynamic matching markets:

AGENTS ALSO ARRIVE ONLINE AND BRING GOODS

- **Unver (2010)** «Dynamic kidney exchange» *RevEconStud,*
- **Bloch, Cantala (2017)** «Dynamic Assignment of Objects to Queuing Agents» *AmerEconJ*
- **Akbarpour, Li, Gharan (2014)** «Dynamic Matching Market Design» *arXiv*
- **Ashlagi, Braverman, Kanoria, Shi (2017)** «Clearing matching markets efficiently: informative signals and match recommendations» *ManagementSci*
- **Ashlagi, Burq, Jaillet, Saberi (2018)** «Maximizing Efficiency in Dynamic Matching Markets» *arXiv*

Computer Science. Fairness without efficiency:

- **Walsh (2011)** «Online cake cutting» *Lect.Notes in CS*
- **Aleksandrov, Aziz, Gaspers, Walsh (2015)** «Online Fair Division: Analysing a Food Bank Problem» *IJCAI*
- **Kash, Procaccia, Shah (2014)** «No Agent Left Behind: Dynamic FD of Multiple Resources» *J.Art.Intell*
- **Benade, Kazachkov, Procaccia, Psomas (2018)** «How to Make Envy Vanish Over Time» *EC-18*

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a by -product:
first exact values of **PoF** for **offline cake-cutting** and **bargaining**

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One random good g is to be allocated to agents $i = 1{,}2\ldots{,}n$

Vector of values $v = (v_i)_{i=1..n} \in \mathbb{R}^n_+$ has arbitrary distribution P

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A Prior-Independent rule φ does not depend* on P

*note that prior free rule «knows» the expected value of v_i

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allocates g to an agent with highest v_i:

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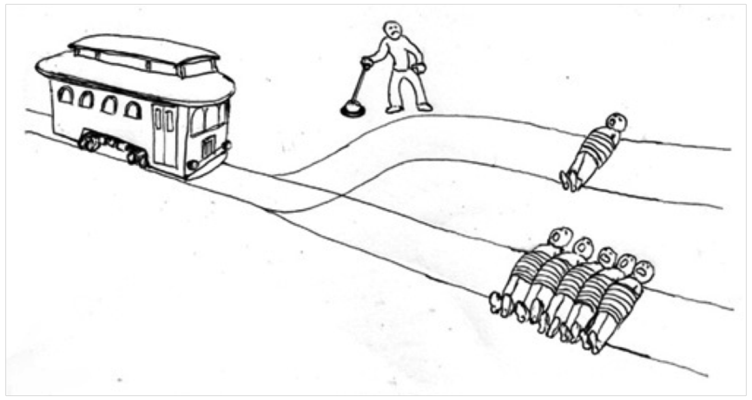

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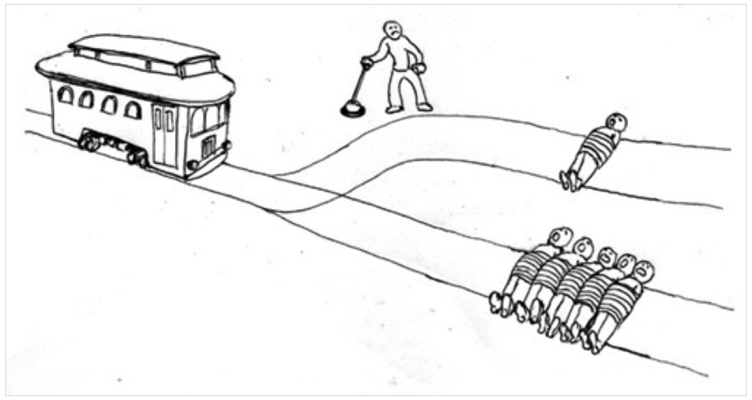

Example:

	p=0.99	p=0.01
v_1	1	1
v_2	1.01	0.01

Agent 1 receives g with probability 0.01 and his expected value $V_1 = \mathbb{E} v_1 \varphi_1(v) = 0.01 \cdot 1 = 0.01$

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Idea of the proof (n=2):

- want to prove $\mathbb{E}\frac{v_1^2}{v_1+v_2} \geq \frac{1}{2}$ and know that $\mathbb{E}v_1 = \mathbb{E}v_2 = 1$
- there is a linear lower bound $\frac{v_1^2}{v_1+v_2} \geq \frac{3}{4}v_1 - \frac{1}{4}v_2$
- take expectation from both sides.

THE MOST EFFICIENT FAIR RULE FOR TWO AGENTS

- **Ex-post welfare domination:**

$$\varphi \succcurlyeq \psi \Leftrightarrow \forall v \quad \sum_i v_i \varphi_i(v) \geq \sum_i v_i \psi_i(v)$$

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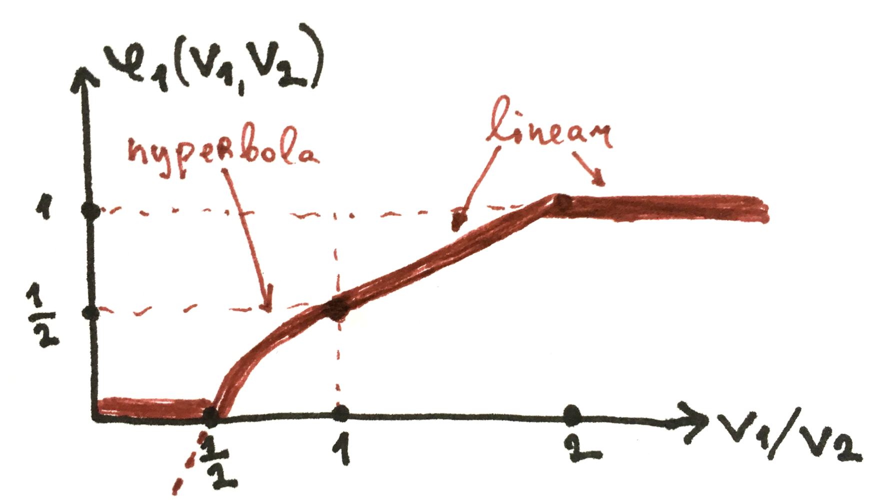
- **The Top-Heavy (TH) rule (n=2):**

$$\varphi_1(v_1, v_2) = 1 - \varphi_2(v_1, v_2) = \begin{cases} 0 & \frac{v_1}{v_2} \leq \frac{1}{2} \\ 1 & \frac{v_1}{v_2} \geq 2 \\ 1 - \frac{1}{2}\frac{v_2}{v_1} & \frac{v_1}{v_2} \in [\frac{1}{2}, 1] \\ \frac{1}{2}\frac{v_1}{v_2} & \frac{v_1}{v_2} \in [1, 2] \end{cases}$$



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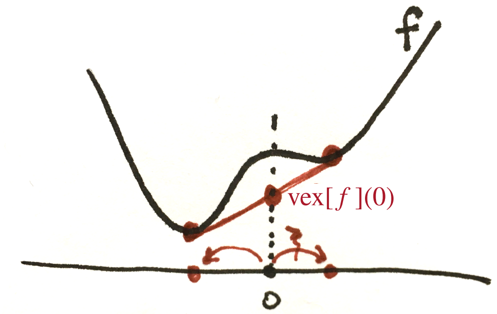

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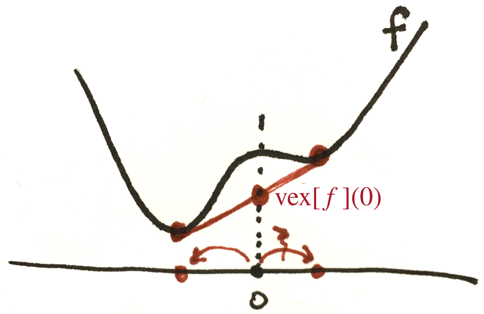

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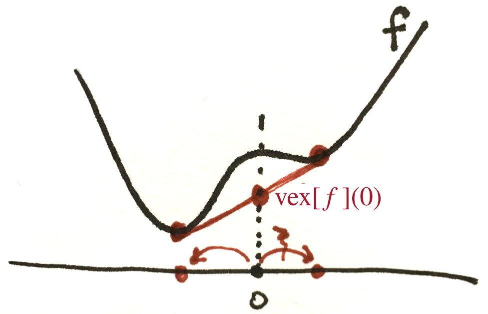

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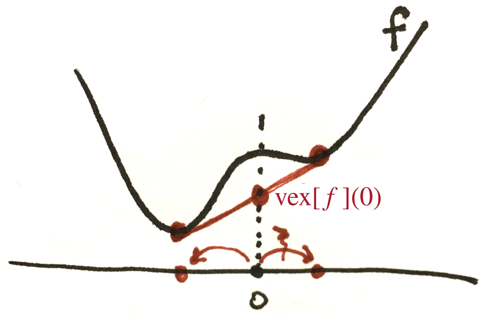

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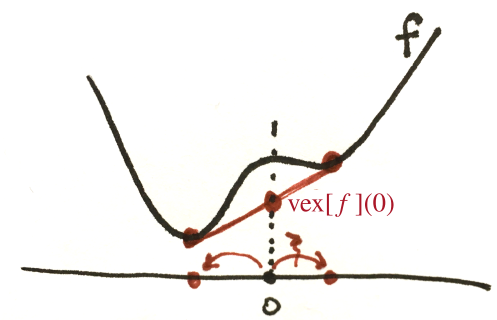

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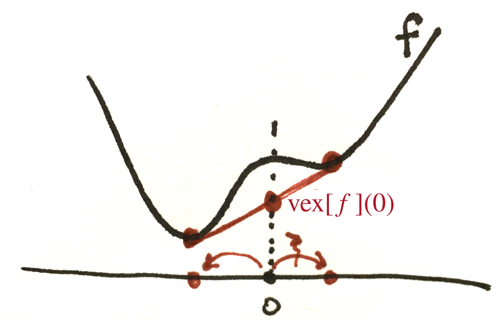

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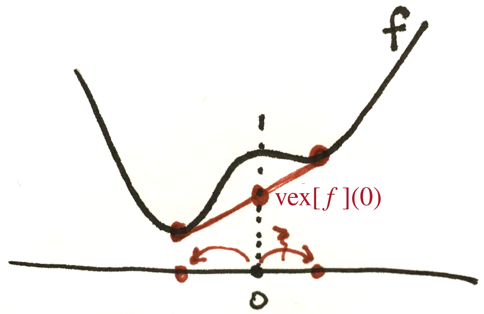

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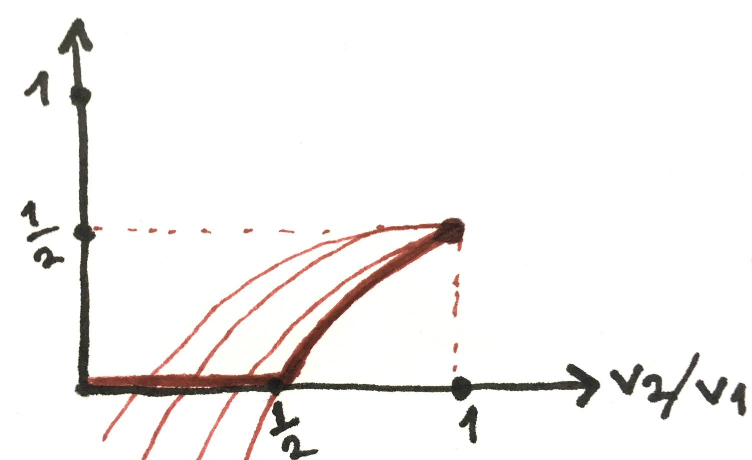

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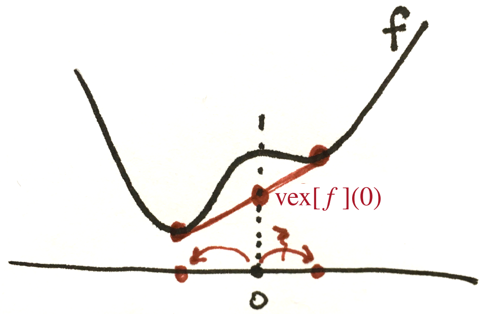

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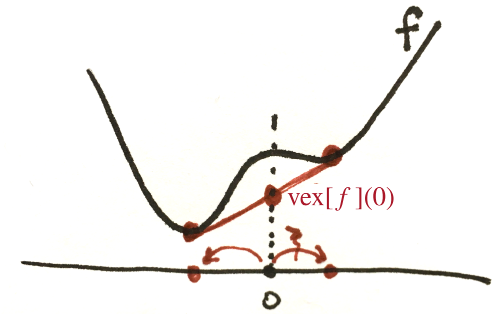

COROLLARY: $\mathbb{E}v_1\varphi_1(v) \geq \frac{1}{2} \Longleftrightarrow v_1\varphi_1(v) \geq \alpha v_1 + \beta v_2 + \gamma, \qquad \alpha + \beta + \gamma \geq \frac{1}{2}$

- Symmetry
- $\varphi_1(v_1, v_2) + \varphi_2(v_1, v_2) = 1$

*

CRITERION OF FAIRNESS

$$\varphi_1(v) \geq \frac{1+\theta}{2} - \frac{\theta}{2}\frac{v_2}{v_1}, \quad \theta \in [0,1]$$

TH RULE: SELECT φ_1 AS SMALL AS POSSIBLE FOR $v_1 < v_2$

- Lower bound decreases in θ for $v_1 < v_2$
- Define $\varphi_1(v) = \left(1 - \frac{1}{2}\frac{v_2}{v_1}\right)_+, \ v_1 < v_2$
- Extend by *

BY THE CONSTRUCTION

TH gives less to the low-value agent than any other fair rule $\Longrightarrow$ domination

MORE THAN TWO AGENTS

- **generalised TH rule:**

$$v_i \neq \max_j v_j \;\Rightarrow\; \varphi_i(v) = \left(\frac{1}{n} + \frac{1}{(n-1)} \left(1 - \frac{\sum_j v_j}{n \cdot v_i} \right) \right)_+$$

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Remark: for bads, the dominating Bottom-Heavy rule is unique.

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Bertsimas et. al (2011) The Price of Fairness.
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$n = 2:$ $\quad \mathrm{PoF} = \sqrt{2} - \frac{1}{2} = 0.914214$

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Proof:

- $\text{PoF} = \rho[TH]$
- $\rho[TH] = \min_{v \in \mathbb{R}^n_+} \frac{\langle v, \varphi(v)\rangle}{\max_i v_i}$
- Painful computations

PRIOR-DEPENDENT RULES

NOW THE RULE KNOWS P

LINKS WITH OFFLINE PROBLEMS: CAKE-CUTTING AND BARGAINING

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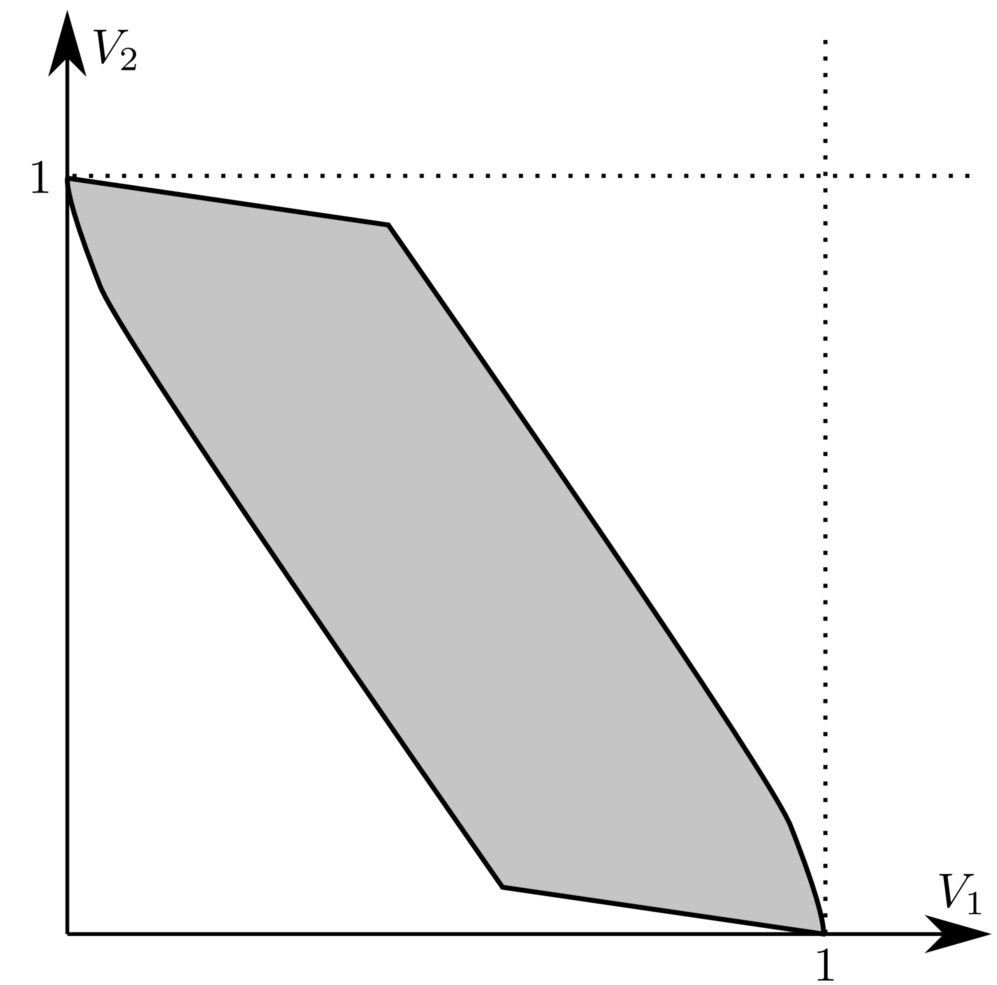

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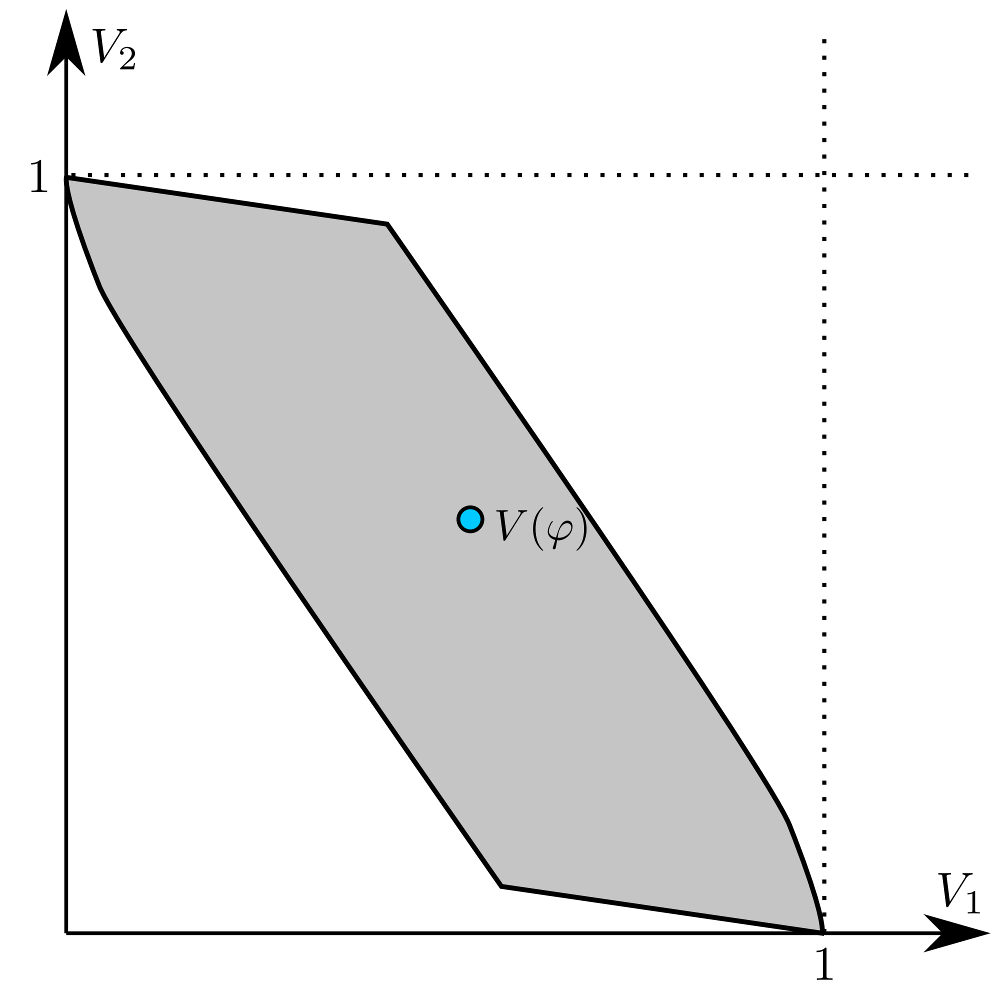

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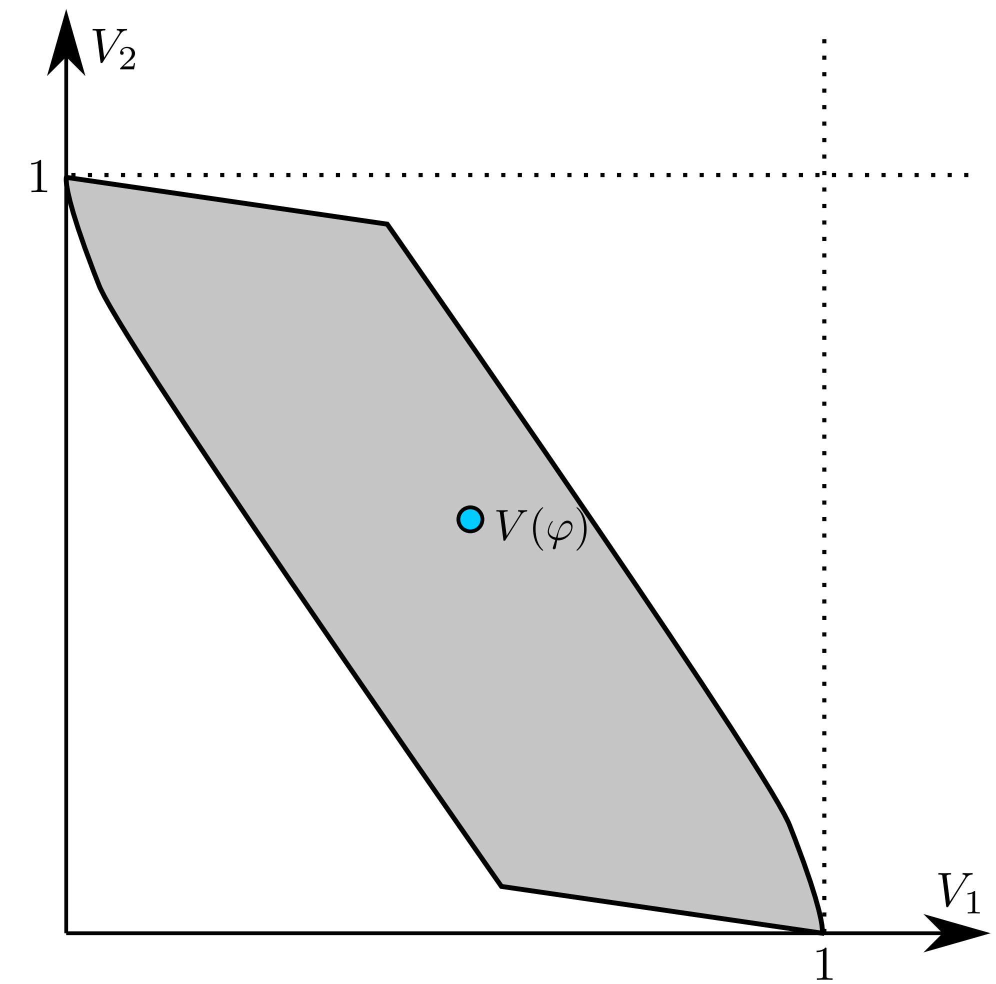

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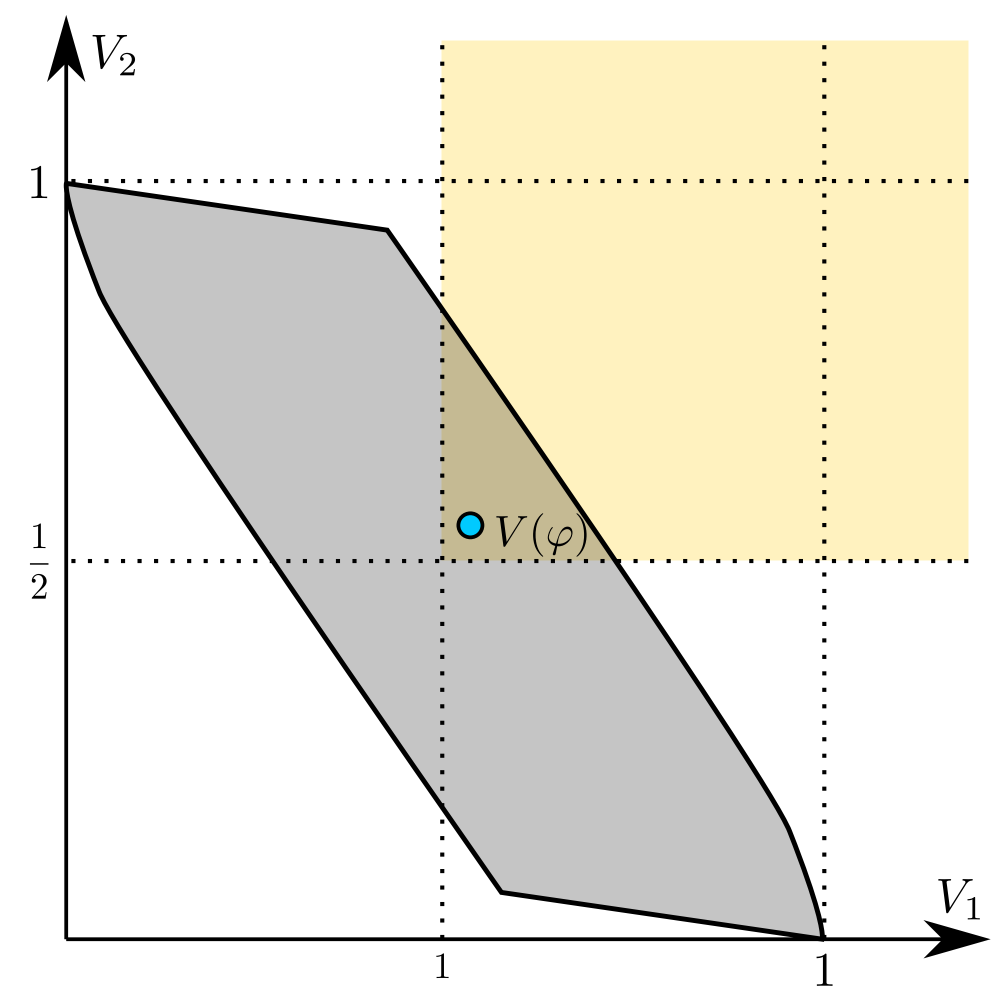

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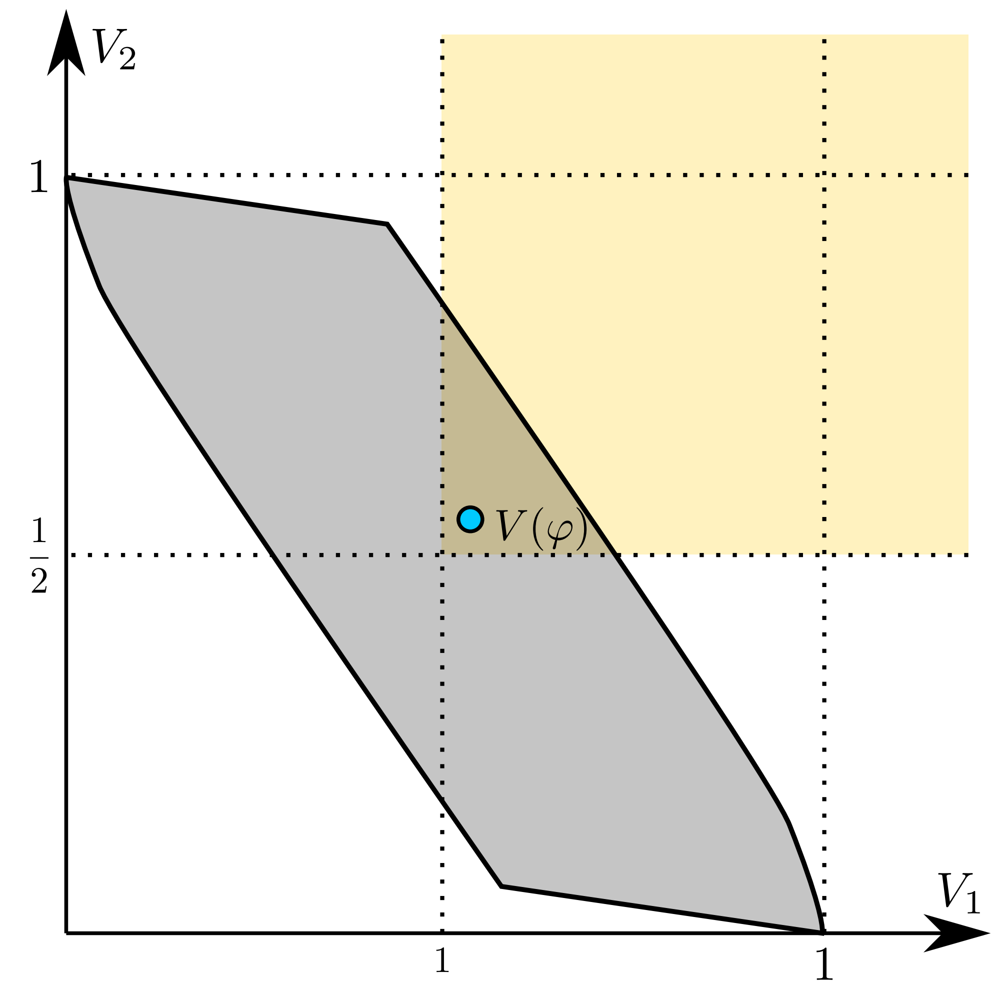

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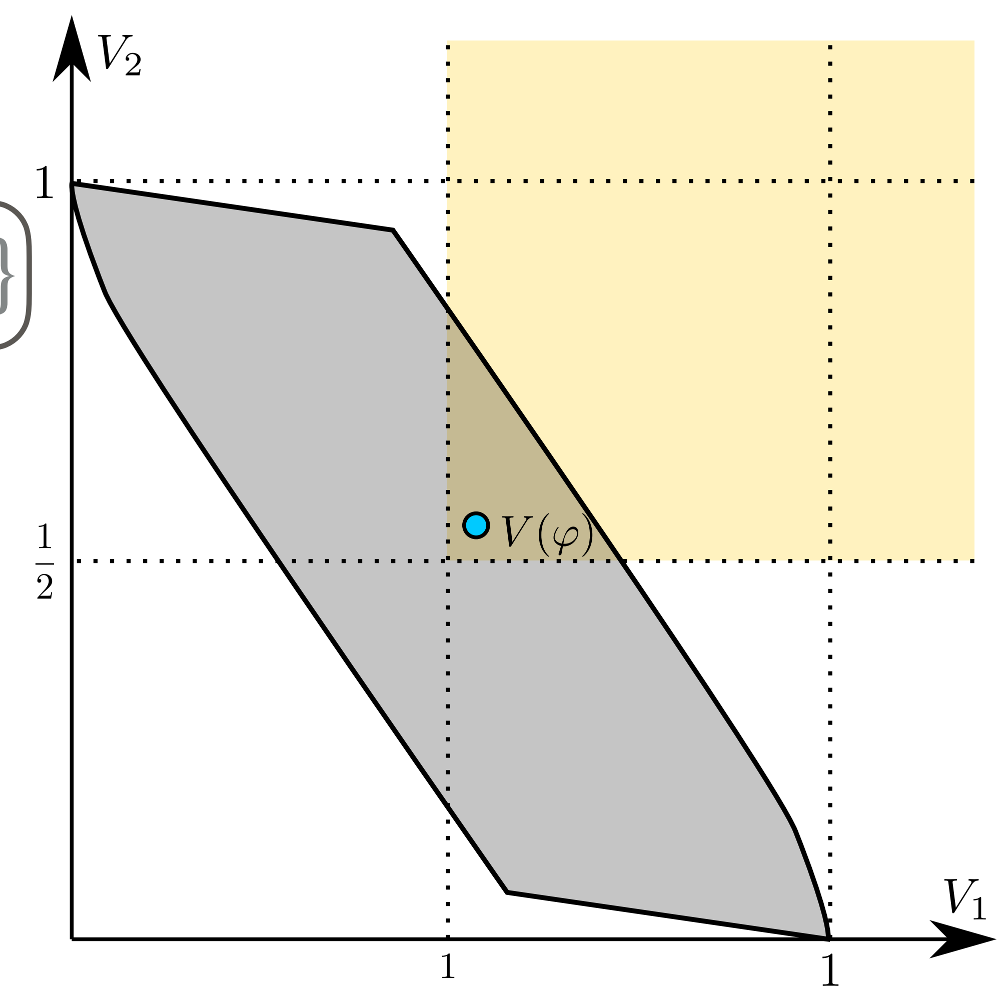

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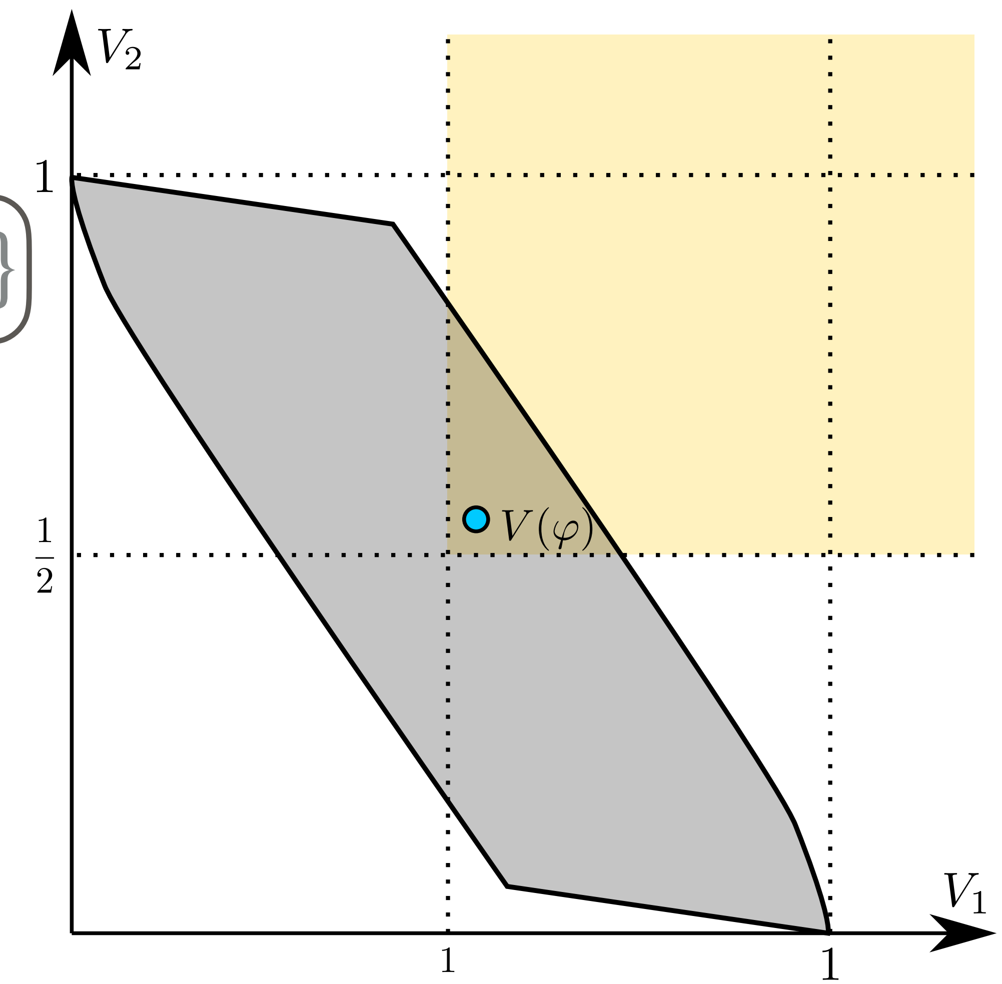

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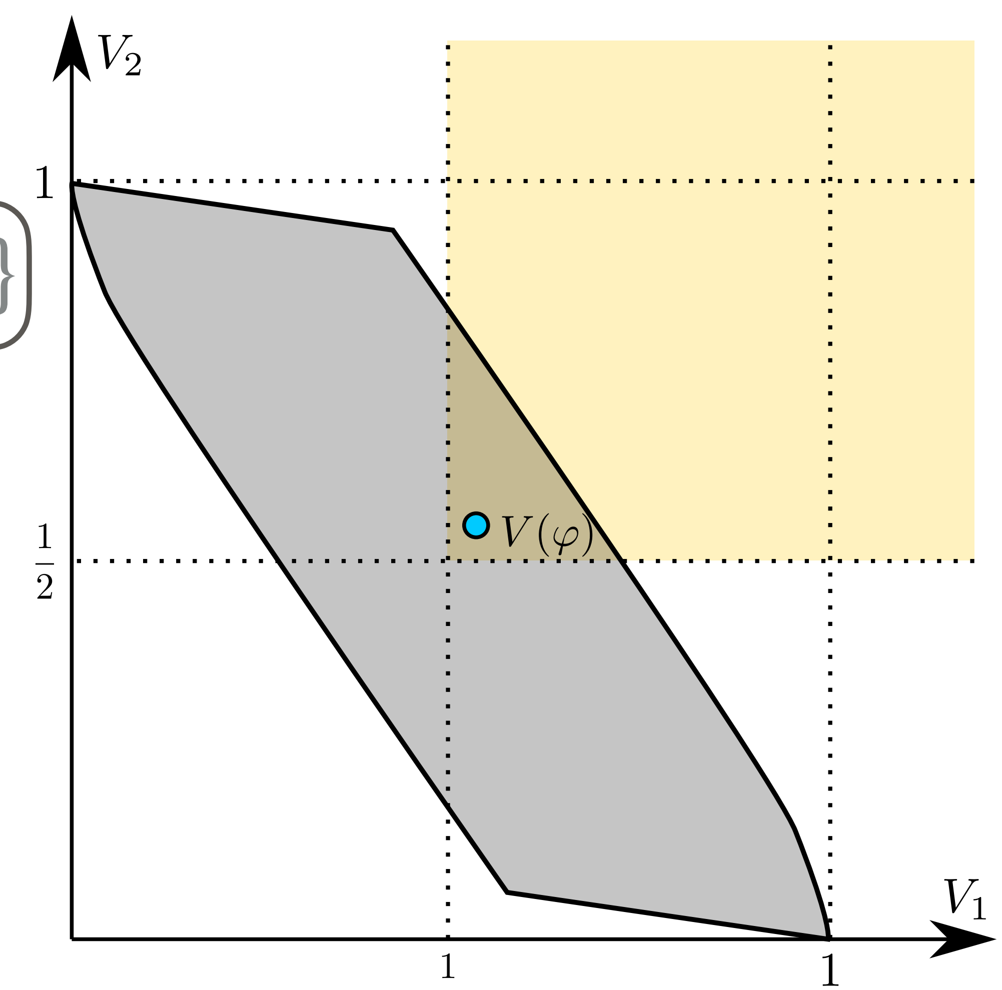

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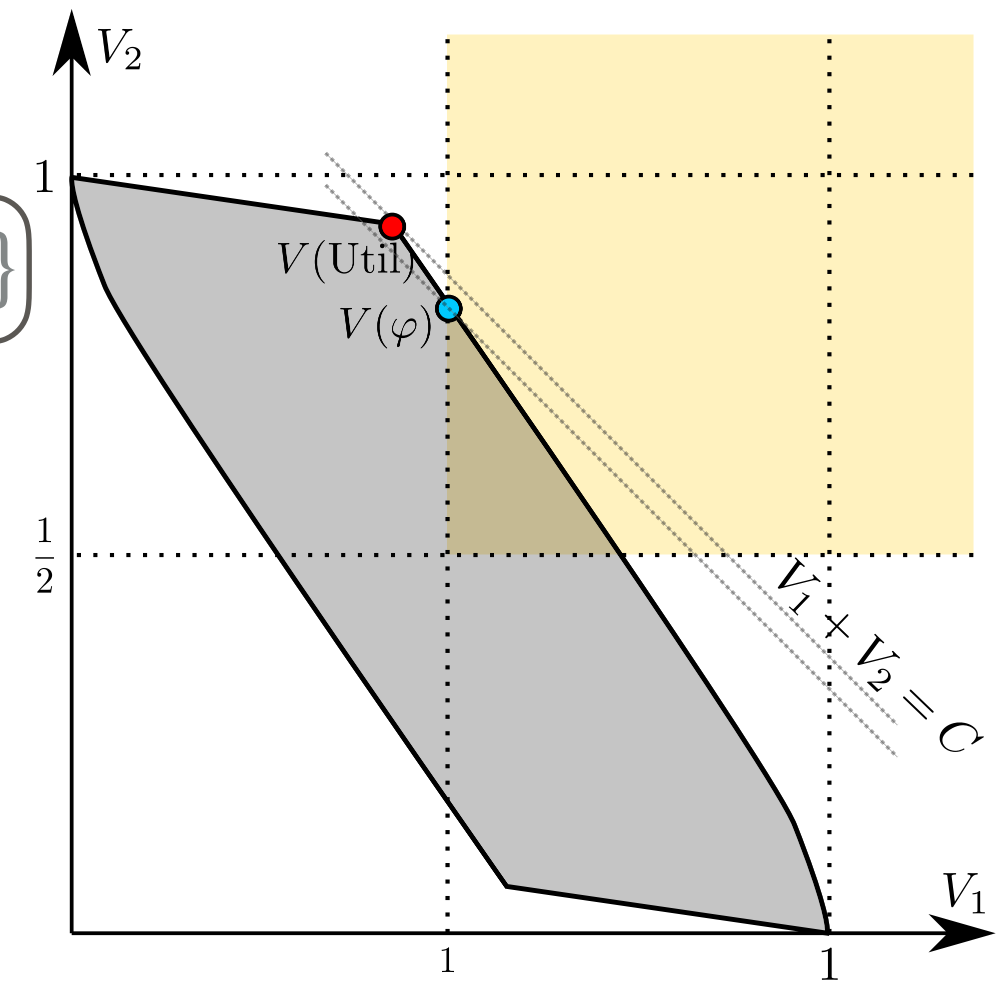

PRICE OF FAIRNESS AND THE MOST EFFICIENT FAIR RULE

THEOREM

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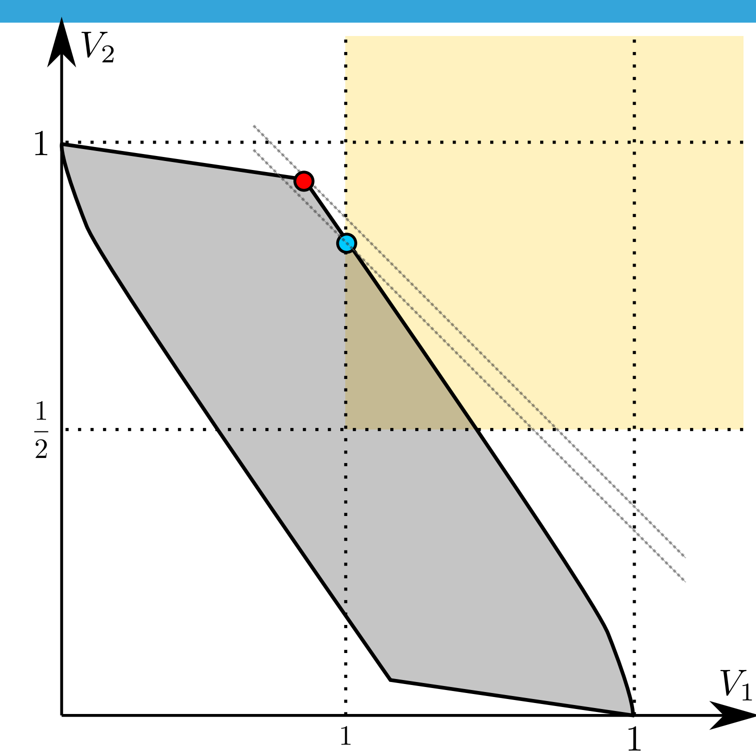

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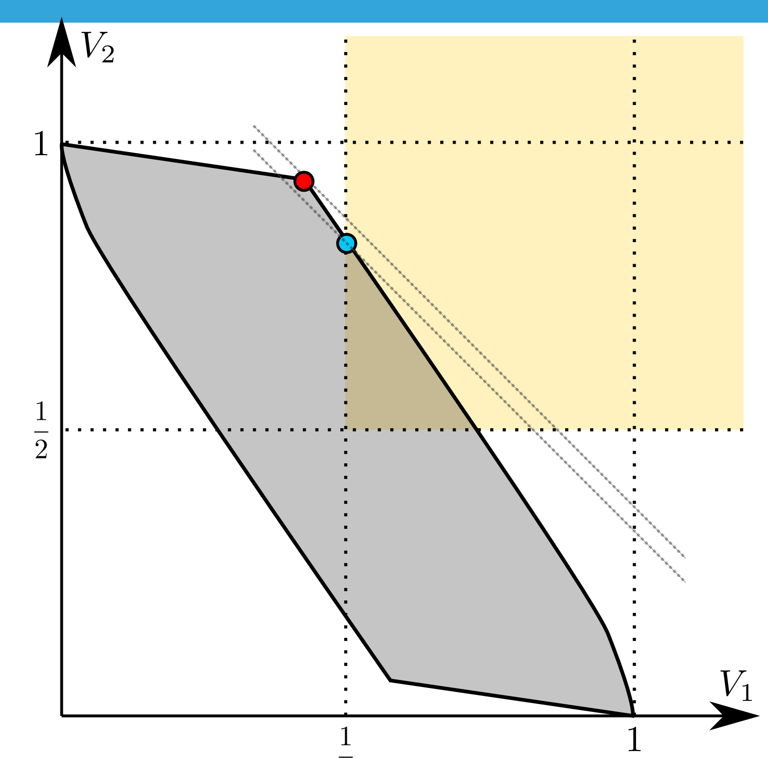

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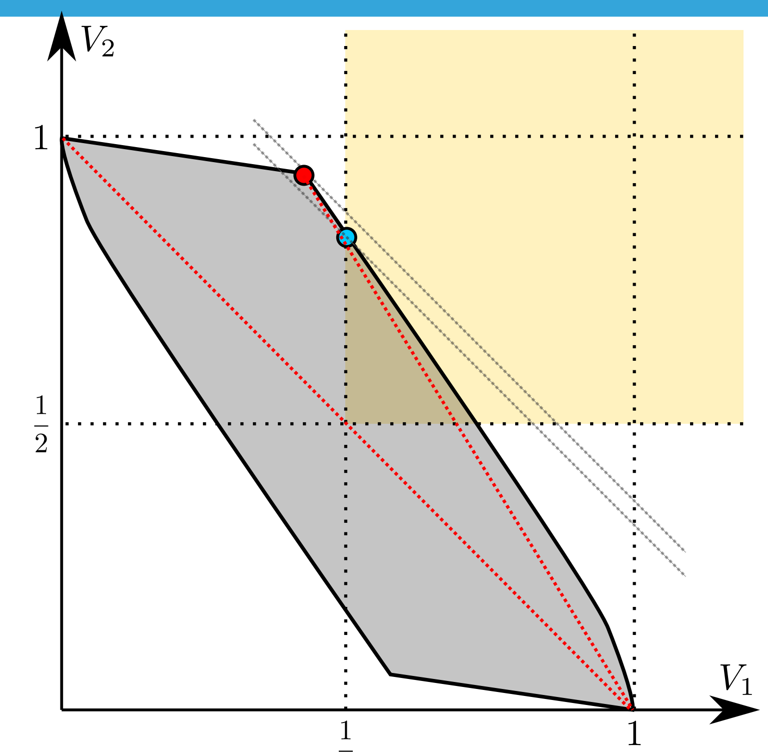

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$$\mathrm{PoF}_{PriorDep} = \mathrm{PoF}_{Cake} = \mathrm{PoF}_{Bargain} =$$

$$\begin{cases} \dfrac{2+\sqrt{3}}{4} = 0.933013 & n=2 \\ \min\limits_{1\le m\le n-1} \{\dfrac{m}{n} + \dfrac{1}{m}\} - \dfrac{1}{n} & n \ge 3 \end{cases}$$

Nash rule: $\prod V_i \to \max$

Bertsimas et al. (2011)

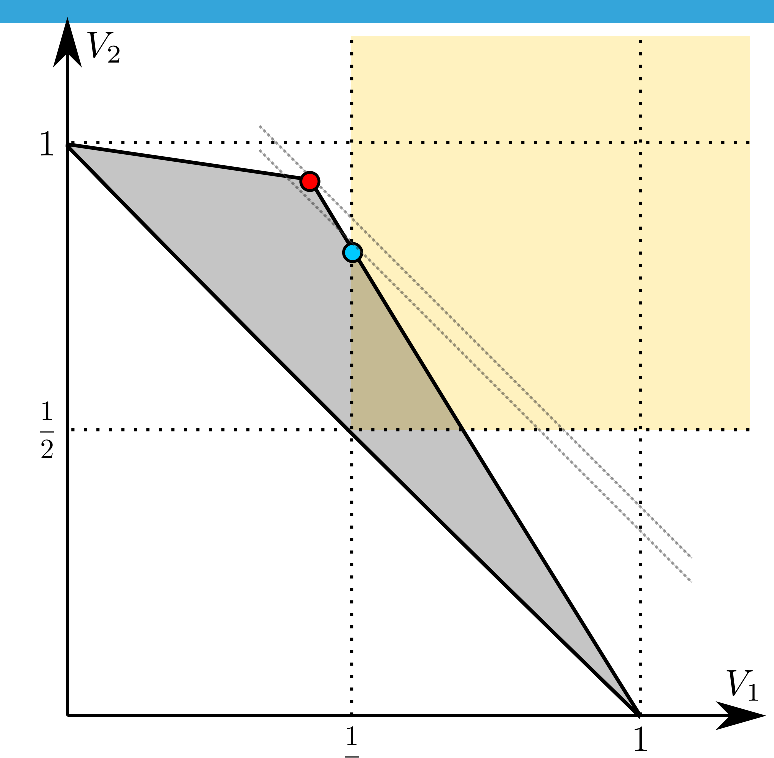
$\rho[\mathrm{Nash}]$: same numbers for $n = k^2$

COROLLARY

Nash rule is the most efficient among fair!

Proof of theorem:

- $\mathrm{PoF}_{Bargain} = \inf\limits_{F} \dfrac{\max_{V\in F\cap\left\{V \ge \frac{1}{n}\right\}} \sum_i V_i}{\max_{V\in F} \sum_i V_i}$
- $\inf\limits_{F} = \min\limits_{F=F(x)}, \quad F(x) = \mathrm{conv}[x, (e_i)_{i=1}^n]$
-



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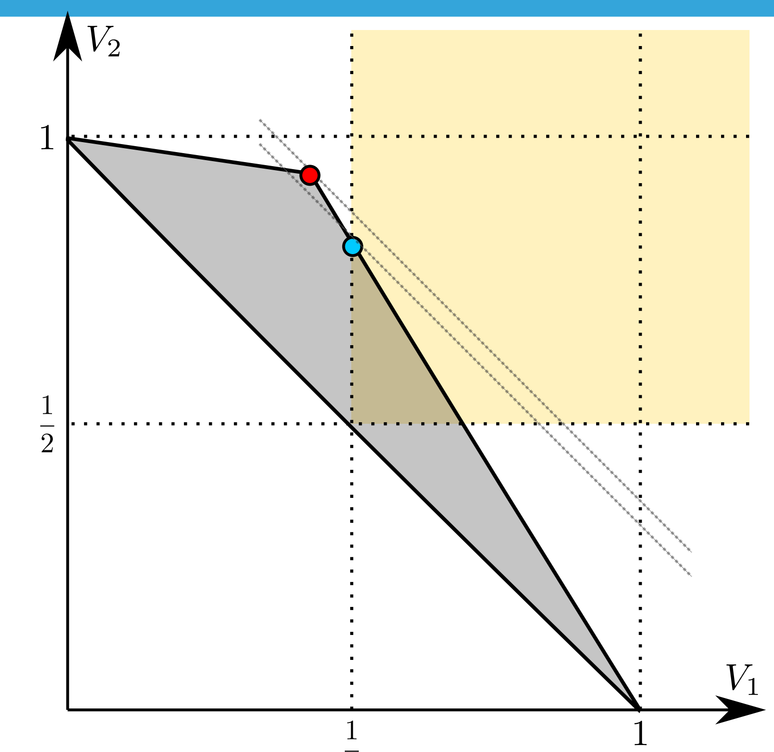
- painful finite-dimensional optimisation



SUMMARY

Prior-Independent: high worst-case efficiency without learning by prior-free mechanisms: simple and robust. Proportional rule is good, TH is the best.

Prior-Dependent: Nash rule has the highest worst-case efficiency

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Thank you! Questions?